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SOME GENERAL CONDITIONS ASSURING $\mathrm{int}A + B = \mathrm{int}(A + B)$

TAMAKI TANAKA
Department of Information Science, Faculty of Science
Hirosaki University, Hirosaki 036, Japan
DAISHI KUROIWA
Department of Mathematics, Faculty of Science
Niigata University, Niigata 950–21, Japan

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Abstract—The aim of this paper is to give a generalization of the authors' result, which showed that the convexity of sets A and B assures $\mathrm{cor}A + B = \mathrm{cor}(A + B)$ and $\mathrm{int}A + B = \mathrm{int}(A + B)$ in a linear space and a linear topological space, respectively.

1. INTRODUCTION

The concept of convexity is very important in various fields of mathematics as well as the area of applied mathematics. The origin of interest in convexity arises from areas of application related to fixed point theory and optimization theory. Recently, we proved such an elementary property of convex sets as in this paper's title. In this paper, we will generalize the authors' results in [3], which shows that the condition $\mathrm{cor}A + B = \mathrm{cor}(A + B)$ holds if A and B are nonempty convex sets with $\mathrm{cor}A \neq \emptyset$ in a linear space, and that the condition $\mathrm{int}A + B = \mathrm{int}(A + B)$ holds if A and B are nonempty convex sets with $\mathrm{int}A \neq \emptyset$ in a linear topological space.

Throughout this paper, the term linear space will refer to a linear space over the real field R or over the complex field C. Given a linear space X, and $a, b \in X, a \neq b$, we will use the following notation for line segment subsets (joining a and b) of X; $[a, b] := \{\lambda a + (1 - \lambda)b : 0 \leq \lambda \leq 1\}$, $[a, b) := [a, b] \setminus \{b\}$, and $(a, b) := [a, b) \setminus \{a\}$. A subset A of X is said to be convex if for every $a, b \in A$, $a \neq b$, the line segment $[a, b]$ is a subset of A. Also, a subset A of X is said to be midconvex if for every $a, b \in A$, $\frac{1}{2}(a + b) \in A$. Of course, any convex set is also midconvex. Let us define addition and scalar multiplication on the family $P(X)$ of nonempty subsets of X by $A + B := \{a + b : a \in A, b \in B\}$ and $\lambda A := \{\lambda a : a \in A\}$, where $A, B \in P(X)$ and λ is a scalar. Also, we will use the following symbols: $x + A := \{x + a : a \in A\}$, where $x \in X$. Moreover, let C be a subset of X, then a subset A of X is said to be C-convex if $A + C$ is a convex set. This concept is usually used in vector optimization and multiobjective programming when the set C is a convex cone.

Next, we define the core operator, interior operator, and the closure operator. For a subset A of a linear space X, the core of A (or algebraic interior of A), written $\mathrm{cor}A$, is the set of all points $a \in A$ such that for each $x \in X \setminus \{a\}$ there exists $b \in (a, x)$ for which $[a, b] \subset A$. For a subset A of a linear topological space $(X, \mathcal{T})$, the interior of A (or topological interior of A), written $\mathrm{int}A$, is defined by $\mathrm{int}A := \cup\{U \in \mathcal{T} : U \subset A\}$. Also, for a subset A of a linear topological space $(X, \mathcal{T})$, the closure of A (or topological closure of A), written $\mathrm{cl}A$, is defined by $\mathrm{cl}A := \cap\{F \subset X : F \supset A \text{ and } F \text{ is closed}\}$. We remark that $\mathrm{cor}A$, $\mathrm{int}A$, and $\mathrm{cl}A$ are convex (or empty) whenever A is convex.

2. ADDITIVITY OF CORE OPERATOR AND INTERIOR OPERATOR

At first, we observe some properties of the core operator and the interior operator without convexity.

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LEMMA 2.1. [4, Proposition 2.1] *Let A and B be nonempty sets in a linear space X. If* $\mathrm{cor}A \neq \emptyset$ *then*

$$\mathrm{cor}A + B \subset \mathrm{cor}(A + B). \tag{2.1}$$

LEMMA 2.2. [1, p. 40] *Let A and B be nonempty sets in a linear topological space X. If* $\mathrm{int}A \neq \emptyset$ *then*

$$\mathrm{int}A + B \subset \mathrm{int}(A + B). \tag{2.2}$$

Next, we cite the following results related to convexity.

LEMMA 2.3. [2, pp. 10 and 59] *The following statements hold:*

(1) *If A is a convex set in a linear space X and $p \in \mathrm{cor}A$, then $[p, a) \subset \mathrm{cor}A$ for any $a \in A$, $a \neq p$;*
(2) *If A is a convex set in a linear topological space X and $\mathrm{int}A \neq \emptyset$, then $t\mathrm{cl}A + (1-t)\mathrm{int}A \subset \mathrm{int}A$ for $0 \leq t < 1$.*

LEMMA 2.4. [2, p. 59] *Let A a set in a linear topological space X. If $\mathrm{int}A \neq \emptyset$ then the following statements hold:*

(1) $\mathrm{int}A \subset \mathrm{cor}A$;
(2) $\mathrm{int}A = \mathrm{cor}A$ *whenever A is convex.*

Now, we prove the main theorem, which is a generalization of Theorem 2.1 in [3].

THEOREM 2.1. *Let A and B be sets in a linear space X. If A is convex with $\mathrm{cor}A \neq \emptyset$ and $\mathrm{cor}A + B$ is midconvex, then*

$$\mathrm{cor}A + B \supset \mathrm{cor}(A + B), \tag{2.3}$$

and hence

$$\mathrm{cor}A + B = \mathrm{cor}(A + B). \tag{2.4}$$

PROOF. For any $x \in \mathrm{cor}(A+B)$, there are $a \in A$ and $b \in B$ such that $x = a+b$. If $a \in \mathrm{cor}A$, then (2.3) holds. Let $a \notin \mathrm{cor}A$. Since $\mathrm{cor}A \neq \emptyset$, there exists a vector $p \in \mathrm{cor}A$, and so $[p, a) \subset \mathrm{cor}A$ by (1) of Lemma 2.3. For $2a - p + b \in X$, $x \in \mathrm{cor}(A + B)$ implies that there exists $0 < \lambda < 1$, such that $x_1 := \lambda(2a - p + b) + (1 - \lambda)x \in A + B$. Let $x_2 := \lambda(p + b) + (1 - \lambda)x$, then we have $x_2 \in [p + b, x) = [p, a) + b$, and hence $x_2 - b \in [p, a) \subset \mathrm{cor}A$. Since there are $\hat{a} \in A$ and $\hat{b} \in B$ such that $x_1 = \hat{a} + \hat{b}$, we have

$$x = \frac{x_2 + x_1}{2} = \frac{x_2 - b + (\hat{a} + \hat{b}) + b}{2},$$

and

$$\frac{x_2 - b + \hat{a}}{2} + \hat{b},\ \frac{x_2 - b + \hat{a}}{2} + b \in \mathrm{cor}A + B$$

by (1) of Lemma 2.3 again. Since $\mathrm{cor}A + B$ is midconvex, we obtain

$$x = \frac{1}{2}\left\{\left(\frac{x_2 - b + \hat{a}}{2} + \hat{b}\right) + \left(\frac{x_2 - b + \hat{a}}{2} + b\right)\right\} \in \mathrm{cor}A + B. \tag{2.5}$$

Thus, (2.3) is proved. So we have (2.4) by Lemma 2.1. ∎

The essential point of the proof of Theorem 2.1 is (2.5).

THEOREM 2.2. *Let A and B be sets in a linear topological space X. If A is convex with $\mathrm{int}A \neq \emptyset$ and $\mathrm{int}A + B$ is midconvex, then*

$$\mathrm{int}A + B \supset \mathrm{int}(A + B), \tag{2.6}$$

and hence,

$$\mathrm{int}A + B = \mathrm{int}(A + B). \tag{2.7}$$

PROOF. Since A is a convex set with $\mathrm{int}A \neq \emptyset$, it follows from (2) of Lemma 2.4 that $\mathrm{int}A = \mathrm{cor}A$. Hence, we have

$$\mathrm{int}(A+B) \subset \mathrm{cor}(A+B) \subset \mathrm{cor}A + B = \mathrm{int}A + B,$$

by (1) of Lemma 2.4 and Theorem 2.1. Thus, (2.6) is proved, and hence, (2.7) follow from Lemma 2.2. ∎

Theorems 2.1 and 2.2 are generalizations of those in [3], respectively.

COROLLARY 2.1. *The following statements hold:*

(1) *Let A and B be sets in a linear space X. If A is convex with* $\mathrm{cor}A \neq \emptyset$ *and B is* $(\mathrm{cor}A)$-*convex, then (2.3) and (2.4) hold.*
(2) *Let A and B be sets in a linear topological space X. If A is convex with* $\mathrm{int}A \neq \emptyset$ *and B is* $(\mathrm{int}A)$-*convex, then (2.6) and (2.7) hold.*

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